Isomorphisms between moduli spaces of parabolic vector bundles

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Let X be a smooth complex projective curve of genus $g \geq 2$. Let $D = \{x_1, \ldots, x_n\} \subset X$.

Definition (Parabolic vector bundle)

A **full flag** quasi-parabolic vector bundle of rank r on (X, D) is a pair $(E, E_\bullet)$ consisting on a rank r vector bundle E on X endowed with a decreasing filtration of the fibre $E|_x$ $\forall x \in D$.

$$E\big|_x = E_{x,1} \supsetneq E_{x,2} \ldots \supsetneq \ldots \supsetneq E_{x,r} \supsetneq E_{x,r+1} = 0$$

A **parabolic vector bundle** is a quasi-parabolic vector bundle $(E, E_\bullet)$ endowed with a set of increasing **parabolic weights** on each parabolic point, $0 \leq \alpha_1(x) < \alpha_2(x) < \ldots < \alpha_r(x) < 1$, called the system of weights of the parabolic vector bundle.

The **parabolic slope** of a rank r parabolic vector bundle is defined as the quotient

$$par\mu(E, E_\bullet) := \frac{\deg(E) + \sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x)}{r}$$

We say that a parabolic vector bundle $(E, E_\bullet)$ is **(semi-)stable** if for any proper subbundle $0 \neq F \subsetneq E$ we have $par\mu(F, F_\bullet)(\leq) < par\mu(E, E_\bullet)$ where $F_\bullet$ denotes the filtration induced by the filtrations of $E_\bullet$ of each fibre $E|_x$ on the subspace $F|_x \subsetneq E|_x$.

Let $M(X, D, r, \alpha, \xi) = M(r, \alpha, \xi)$ denote the moduli space of full flag semistable parabolic vector bundles $(E, E_\bullet)$ of rank r on (X, D) with parabolic system of weights α and $\det(E) \cong \xi$.

Definition (Stability chambers)

If $(E, E_\bullet)$ of rank r and degree d is strictly α-semistable then there exists F of rank $r' < r$ and degree d' such that

$$\frac{d' + \sum_{x\in D}\sum_{i\in I_F(x)}\alpha_i(x)}{r'} = \frac{d + \sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x)}{r}$$

so, taking $n_i(x) = 1$ if $i \in I_F(x)$ and $n_i(x) = 0$ otherwise, we have

$$r\sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x) - r'\sum_{x\in D}\sum_{i=1}^{r}n_i(x)\alpha_i(x) = rd' + r'd$$

This equation defines a set of hyperplanes in the **stability space** $\Delta \subset [0,1)^{(r-1)|D|}$ called **stability walls**. A parabolic weight is **generic** if it does not belong to any of these walls. A numerical wall is called a **geometric wall** if there exists a vector bundle E and a subbundle F realizing the equation. Geometric walls divide the stability space in **stability chambers**. Let C_α be the chamber of α.

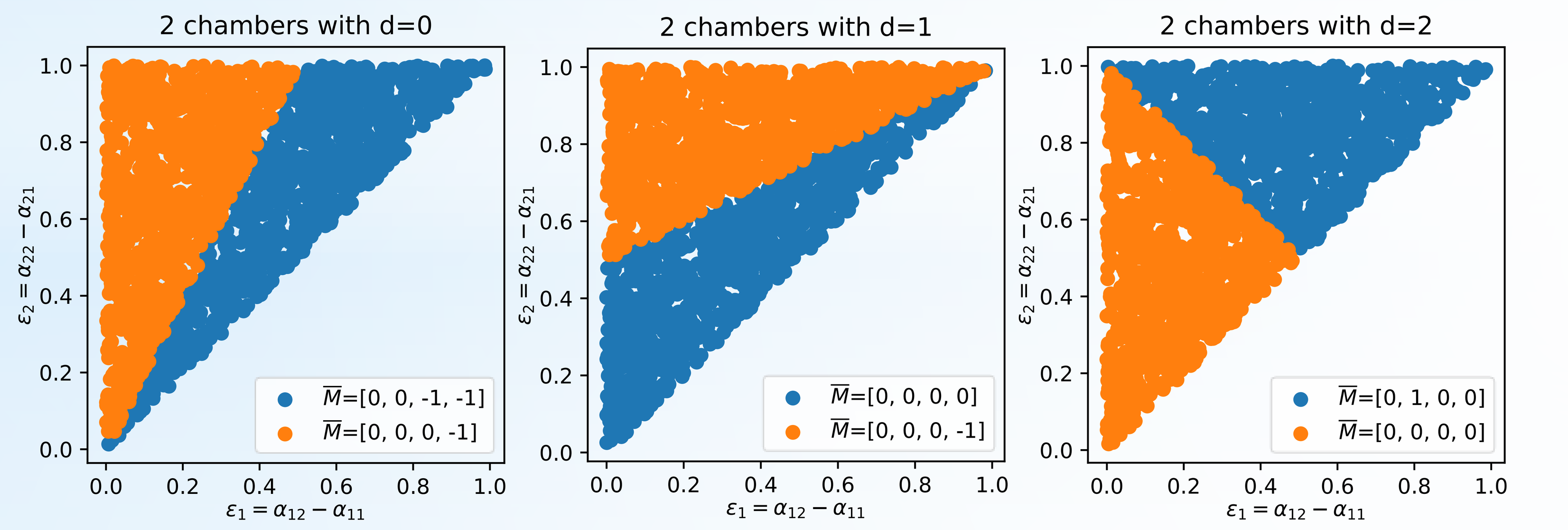
Define a **selection vector** as $\bar{n} = \{n_1(x), \ldots, n_r(x)\}$, with $n_i \in \{0,1\}$ and $\sum_{i=1}^{r} n_i(x) = r' < r$, and let $d = \deg(\xi)$. With all possible $\bar{n}$ we form the vector:

$$\bar{M}(r, \alpha, d) = \left(\left\lfloor \frac{r'd + r'\sum_{x\in D}\sum_{i=1}^{r}\alpha_i(x) - r\sum_{x\in D}\sum_{i=1}^{r}n_i(x)\alpha_i(x)}{r}\right\rfloor\right)_{\bar{n}}$$

By [AG21] we know that $\bar{M}(r, \alpha, d)$ is a **stability chamber invariant**.

Example: Stability walls and chambers for $|D| = 2$, $r = 3$

With above's result, one can use a Montecarlo brute force computation to find different stability chambers. Sample a sufficiently large number of α and calculate the invariant $\bar{M}(r, \alpha, d)$ for each of them.



Each $\bar{n}$ corresponds to a hyperplane normal vector. Only a finite number of intercepts generate intersecting hyperplanes. Let $u_{\bar{n}}$, $l_{\bar{n}}$ be the upper and lower bounds for the possible intercepts.

Lemma (Number of stability walls for r prime)

The number of different **stability walls** *for rank r, degree 0 with n parabolic points is*

$$\begin{cases} \frac{1}{2}\sum_{r'=1}^{\lfloor r/2\rfloor}\binom{r}{r'}^{n-1}\left(nS_{r'} - \binom{r}{r'}\right) & \text{if } r > 2, r \text{ prime} \\ nr^{n-1} & \text{if } r = 2 \end{cases} \quad \text{with } S_{r'} = \sum_{\bar{n}_{r'}} u_{\bar{n}} - l_{\bar{n}}$$

Note: A similar but more complex equation was found for all non-prime r.

Find below to the left the exact number of stability walls for each $n = |D|$ and rank r. Note that the number of walls is independent on the degree d. On the right is the exact number of stability chambers for each $n = |D|$, rank r and degree d, computed using a deterministic and exact algorithm described in the following section.

Result (# stability walls)

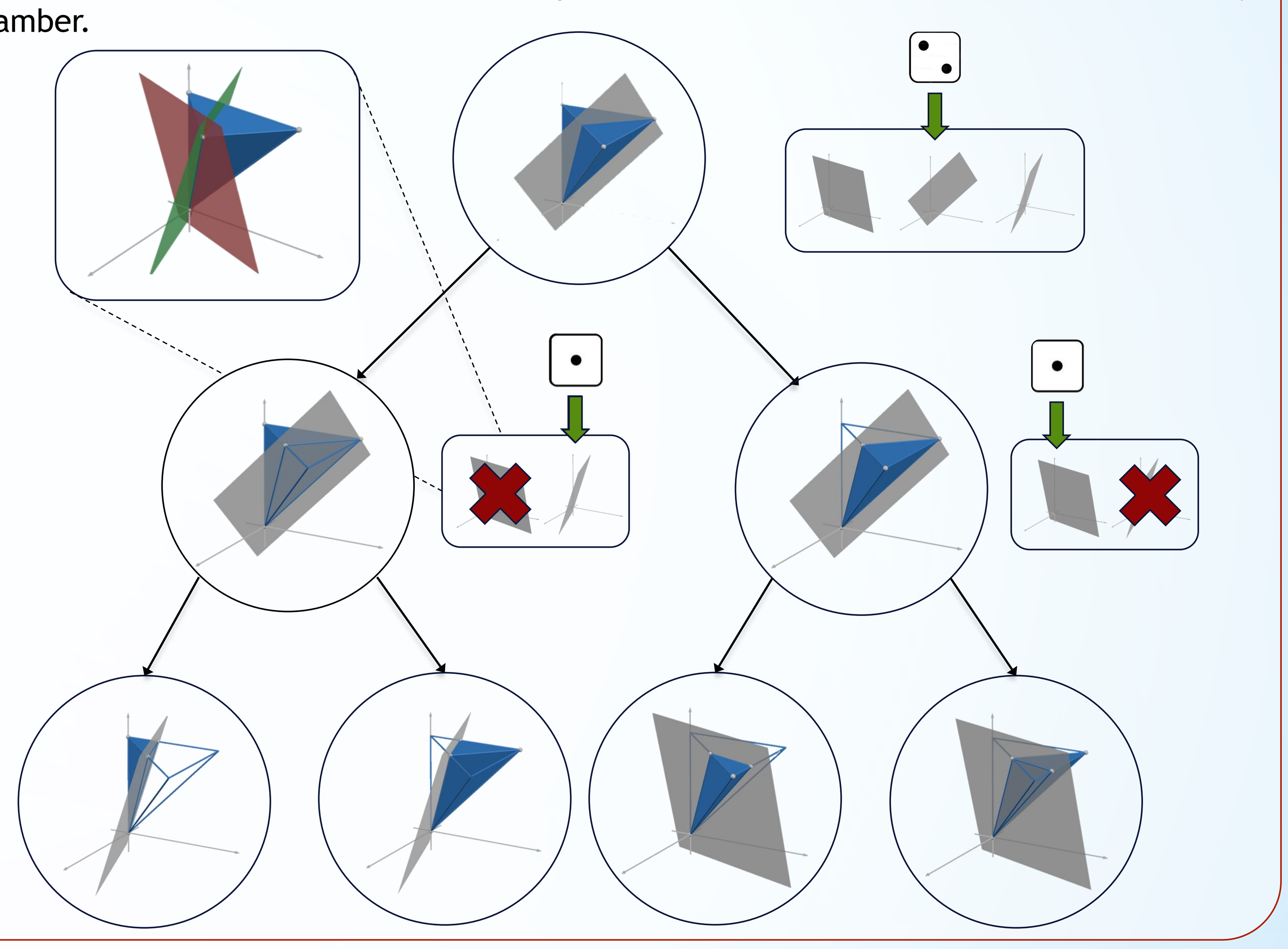
	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9
n=1	0	1	3	11	21	65	129	307
n=2	1	9	41	215	799	3927	15049	65403
n=3	4	45	344	3075	21379	186837	1354856	11062215
n=4	12	189	2540	39875	515229	8200787	112008812	
n=5	32	729	17840	491875	11827979	345458281		
n=6	80	2673	122384	5871875	264528629			
n=7	192	9477	828416	68421875				
n=8	448	32805	5555648	782421875				

Result (# stability chambers)

	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9
n=1	1	2	4	14	80	1296	76724	>5315121
n=2	2	12	640	4748330				
n=3	5	720	>1984886					
n=4	24	4868610						
n=5	409							
n=6	31916							
n=7	10834621							

Algorithm (Binary tree classification)

We devised an algorithm based on binary trees that **identifies all different stability chambers** for any $|D|$, rank r and degree d. The idea is to start with the set of all hyperplanes that intersect the the stability space Δ, divide the region with one of them chosen randomly, and then for each of the children's regions, filter our hyperplanes that don't intersect with it. Once no more planes are left the current node is a stability chamber.



Definition (Basic transformation)

A basic transformation on (X, d) is a tuple $T = (\sigma, s, L, H)$ where

- σ *is and automorphism* $\sigma: X \to X$ such that $\sigma(D) = D$
- $s \in \{-1, 1\}$, *which corresponds to taking the dual of the quasiparabolic vector bundle*
- $L \in Pic(X)$ is a tensorization
- H is a divisor on X with $0 < H < (r-1)|D|$.

Basic transformations acts on families of quasiparabolic bundles as follows

$$T(E, E_\bullet) = \begin{cases} \sigma^*(L \otimes \mathcal{H}_H(E, E_\bullet)) & s = 1 \\ \sigma^*(L \otimes \mathcal{H}_H(E, E_\bullet))^\vee & s = -1 \end{cases}$$

Where $\mathcal{H}_H$ denotes the Hecke transformation of $E_\bullet$ made at each parabolic point the number of times indicated by the divisor H.

All isomorphisms between $M(X, D, r, \alpha, \xi)$ are induced by compositions of these basic transformations and there is a finite number of them [AG21].

Algorithm (Automorphism graph)

We devised an algorithm that find all **isomorphism classes** and their corresponding **list of automorphisms**. Idea: Start with the classification tree mentioned above. Iterate through each unvisited leaf node, with an assigned representative α. For all basic transformations T that don't alter the degree d, compute $\alpha' = T(\alpha)$, run it through the classification tree and mark the outputted node as visited. If $C_{\alpha'} = C_\alpha$, add T to automorphism list.

Result (# isomorphism classes)

	r=2	r=3	r=4	r=5	r=6	r=7
n=1	1	1	2	7	40	648
n=2	1	2	44	237818		
n=3	2	17				
n=4	3	4782				
n=5	8					
n=6	28					
n=7	270					

After computing the automorphism groups for all isomorphism classes of moduli spaces of parabolic bundles with small r and small $n = |D|$ we find:

- Many types of theoretically possible automorphisms can never happen in a generic weight.
- Dual never is a part of an automorphism.

Theorem (Dual breaks automorphisms)

If $T = (\sigma, s, L, H)$ is a basic transformation inducing an automorphism of the moduli space $M(X, D, r, \alpha, \xi)$, then $s = 1$.

Example: Automorphism stats for $|D| = 6$, $r = 2$

As mentioned above, we found the automorphism groups for all isomorphism classes of moduli spaces of parabolic bundles with small rank and small number of parabolic points. One interesting statistic is the number of isomorphisms classes with a specific number of automorphisms. Note that the automorphisms are counted modulo the 2^{2g} possible tensorizations by r-torsion line bundles and the symmetries of the curve which fix each parabolic points $|Aut(X; x_1, \ldots, x_n)|$

# of automorphisms	8	16	24	48	72	96	240	288	1440	3840	Total
# of isomorphism classes	2	1	5	3	3	4	4	2	3	1	28

References

[AG21] D. Alfaya, T.L. Gómez, Automorphism group of the moduli space of parabolic bundles over a curve. Advances in Mathematics. Vol. 393, pp. 108070, 2021.

[AHPPR] D. Alfaya, S. Herreros, J. Pizarroso, J. Portela, J. Rodrigo. Isomorphisms between moduli spaces of parabolic vector bundles. *In preparation*

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